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A variational principle for the statistical mechanics of fully developed turbulence

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Abstract. The stationary equilibrium of turbulent flow is a balance between the external energy input and internal viscous dissipation. We introduce, in a natural way, a maximum chaoticity principle for turbulent equilibrium that leads to the prediction of correct energy spectra in a number of different dynamical situations. The principle has been applied to three- and two-dimensional turbulence, intermittency corrections, thermal convection, helicity turbulence and three- and two-dimensional magnetohydrodynamics. Various analogies have been pointed out with the statistical mechanics of conservation systems to which a maximum entropy principle applies.

1. Introduction

Let us consider a thermodynamic system at thermal equilibrium with a heat reservoir, for which Liouville's theorem holds

$$\frac{\partial}{\partial t}\rho + \sum_i \frac{\partial}{\partial q_i}\rho \dot{q}_i = 0 \tag{1.1}$$

where $\{q_i\}$ are the phase variables of the system and $\rho(\{q_i\}, t)$ is a density function in phase space Γ.

The constraint of probability conservation implies that the stationary probability distribution $F(\mu)$, on the basis of an ergodic postulate, must be a function of additive, first integrals of the motion. If $F(\mu)$ depends on energy only, Gibbs' distribution holds for a 'small' system in equilibrium with a much larger one

$$F(\mu) \propto \exp(-\beta E(\mu)) \tag{1.2}$$

as may also be seen by using arguments based on the central limit theorem (Khinchin 1949).

Gibbs' distribution itself may be obtained by using a maximum entropy principle in the search for the functional $F(\mu)$ describing the equilibrium distribution. We can make the statistical mechanics of an inviscid (zero viscosity) fluid in the usual manner (Salmon *et al* 1976) by decomposing the velocity field $\boldsymbol{V}(\boldsymbol{x}, t)$ into a Fourier series and truncating at a sufficiently high value of k:

$$V_\alpha(\boldsymbol{x}, t) = \frac{1}{L^{d/2}} \sum_{|\boldsymbol{k}|<k_{\max}} V_\alpha(\boldsymbol{k}, t) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) \tag{1.3}$$

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(for the sake of simplicity let us consider a fluid confined in a d-dimensional box of side L, with periodic boundary conditions). For $V_\alpha(\boldsymbol{k}, t)$ variables Liouville's theorem holds at zero viscosity and the energy

$$E=\frac{1}{2}\sum_{\boldsymbol{k},\alpha}|V_\alpha(\boldsymbol{k}, t)|^2$$

is conserved. Gibbs' distribution (1.2) applies to inviscid truncated models of Navier–Stokes (NS) equations leading to energy equipartition among all considered normal modes $V_\alpha(\boldsymbol{k})$. One could naively think that in the case of fully developed turbulence, where the energy is conserved in mean by the external input, equilibrium statistical mechanics holds and the results obtained in the inviscid case are correct. This implies that the energy spectrum $E(k)$ for d-dimensional turbulence has the form

$$E(k)\propto k^{d-1}. \tag{1.4}$$

Equation (1.4) is in violent disagreement with the phenomenological theories of turbulence (Rose and Sulem 1978) which give

$$E(k)\propto\begin{cases}k^{-5/3} & d=3\\ k^{-3} & d=2\end{cases} \tag{1.5}$$

(in the inertial range). Thus the maximum entropy principle does not describe fully developed turbulence even qualitatively because of its substantially 'static' character (it applies to conservative systems for which external energy injection is not required to maintain a stationary state).

On the other hand turbulence dynamics is strongly characterised by such phenomena as energy transfer from one scale of the motion to another, which have no counterpart in ordinary statistical mechanics.

More precisely, if we supply energy from outside to a turbulent system at a constant rate, the system may well attain a stationary condition. Viscous dissipation (which is viscosity independent at high Reynolds numbers (Batchelor 1953)) will indeed remove (by transforming it into heat) supplied energy at a rate identical to that of injection.

The attainment of a stationary state is, however, a basic postulate for any statistical description of turbulence to have a definite sense. However for a turbulent system it is not possible to derive the Gibbs canonical distribution $F(\mu)\propto\exp(-\beta E)$ on phase space for a small part of the system because the whole system does not conserve its energy.

Other maximum randomness principles were proposed to describe the stationary equilibrium of turbulent flows. Fujisaka and Mori (1979) have recently estimated the correction to the Kolmogorov law due to intermittency, postulating that Shannon's entropy of the probability of having intermittent volume must be maximum. This approach can only be used to estimate the intermittent corrections and is always based on the idea of 'equilibrium'. Another maximum randomness principle was proposed by Edwards and McComb (1969) to select the statistical properties of the stationary state. Unfortunately the calculations involved in the use of their principle are very long and difficult because they try to derive a theory from the fundamental equations. In this paper we show that if we state the principle for which the 'predictability time' (Lorenz 1969, 1972, Lilly 1972a, b), i.e. the time in which a perturbation on the smallest scale entering in the motion (the dissipative scale) influences the large scale motion, must be a minimum, we obtain the correct laws for the energy spectrum $E(k)$.

The physical significance of this principle is quite evident: the statistical properties of the system (particularly, as we shall see, the typical velocity fluctuations at different scales of motion) are those which achieve the maximum interaction velocity among the various scales of the fluid motion, that is, the more dynamically chaotic situation.

The fundamental difference between our variational principle and the other ones (Fujisaka and Mori 1979, Edwards and McComb 1969) is its dynamical nature: the previous maximal randomness principles concern the 'static' properties of a stationary state while ours concerns the evolution of a perturbation around a stationary state, i.e. a dynamical property.

In § 2 we shall describe the variational principle and compare our result for $E(k)$ with the Kolmogorov theory in three dimensions. In § 3 we briefly apply our method to several cases of fully developed turbulence. In § 4 we state some analogies with the statistical mechanics. Conclusions are reported in § 5.

2. The variational principle

We shall now give a mathematical expression for the predictability time. For the sake of simplicity let us exponentially divide (Frisch *et al* 1978) the scales of the motion, that is, let us consider the scales $l_1, l_2, \ldots, l_n, \ldots$ where

$$l_n = b^{-n} l_0 \qquad b > 1 \tag{2.1}$$

and l_0 is the 'external length', i.e. a length which is characteristic of the global flow (for example, the correlation length of the velocity field). The predictability time is defined as follows

$$T_p = \sum_{n=1}^{N} \tau_n \tag{2.2}$$

where τ_n is the time in which a perturbation on the nth scale influences the $(n-1)$th scale. N is the order number of the dissipative scale η: the last scale involved in the motion, and for which the Reynolds number is of the order of one

$$\eta = l_N \qquad l_N V_N / \nu \sim 1.$$

Following Lorenz (1969, 1972), Lilly (1972a, b), Orszag (1977) and Frisch (1980) one can believe τ_n to be the typical evolution time of the nth scale structure (the turnover time, characteristic for distortion at scale $\sim l_n$)

$$\tau_n = l_n / V_n = 1/k_n V_n \qquad k_n = l_n^{-1} \tag{2.3}$$

where V_n is the typical velocity difference at scale l_n:

$$V_n^2 = \int_{k_n}^{k_{n+1}} E(k)\,\mathrm{d}k \qquad E(k_n) \sim V_n^2 k_n^{-1}. \tag{2.4}$$

$E(k)$ is the energy spectrum, defined as the kinetic energy for unit mass and unity wavenumber ($\int_0^\infty E(k)\,\mathrm{d}k = \frac{1}{2}\langle V^2 \rangle$). The choice (2.3) comes in a natural way by the Navier–Stokes equations in spectral form if one assumes that the interactions between different scales of motion are essentially local in k space. The predictability time defined in (2.2) is the time in which the system 'forgets' its initial conditions.

The rate of loss of information (in the information theory meaning) about the state of the system may be connected with the deformation of a small representative volume in phase space (Brillouin 1962). More particularly, a parameter that gives a quantitative measure of the sensitive dependence on initial conditions is the maximum Lyapunov number λ_M (for a mathematical definition see Benettin *et al* (1976), Mori and Fujisaka (1980)). In a 'chaotic' system (as a turbulent flow) λ_M is connected with the growth of the distance $D(t)$ in phase space between two points whose initial separation was $D(0)$ (Ruelle 1980a)

$$D(t) \sim D(0) \exp(\lambda_M t). \tag{2.5}$$

λ_M^{-1} clearly represents the time over which it is practically impossible to follow the dynamical evolution of the system.

From our definition of the predictability time it seems possible to state the correspondence

$$T_p \sim \lambda_M^{-1}. \tag{2.6}$$

The inverse of the predictability time appears to be connected to the mean rate of entropy production (in the information theory meaning) (Brillouin 1962, Zaslavskii and Chirikov 1972, Rabinovich 1978).

We are now seeking an expression of V_n that minimises the predictability time. The minimisation of T_p must be constrained by the conservation of the mean energy E of the flow

$$E = \tfrac{1}{2} \sum_n V_n^2 \tag{2.7}$$

(indeed we are interested in the properties of the turbulent motion in the stationary state and far from the dissipative scales).

By making use of the standard technique of Lagrange multipliers we obtain

$$\frac{\partial}{\partial V_n}(T_p + \Lambda E) = 0 \tag{2.8}$$

$$\frac{1}{k_n V_n^2} = \Lambda V_n \tag{2.9}$$

from which

$$V_n = \Lambda^{-1/3} k_n^{-1/3}. \tag{2.10}$$

From the equation (2.10) we directly obtain the spectrum

$$E(k_n) \sim V_n^2 k_n^{-1} = \Lambda^{-2/3} k_n^{-5/3}. \tag{2.11}$$

We easily can see that the Lagrange multiplier Λ is proportional to the inverse of the energy dissipation $\langle \varepsilon \rangle$, indeed

$$\langle \varepsilon \rangle = \nu \sum_n k_n^2 V_n^2 \simeq \nu k_N^2 V_N^2 \tag{2.12}$$

and considering that

$$V_N / k_N \nu \sim 1 \tag{2.13}$$

it follows by (2.10), (2.12) and (2.13) that

$$\Lambda \sim \langle \varepsilon \rangle^{-1}. \tag{2.14}$$

Equations (2.11) and (2.14) constitute the well known '$-5/3$' Kolmogorov law (Monin and Yaglom 1975).

3. Applications of the variational principle

In this section we present the results that one can obtain applying the variational principle seen in § 2 to the turbulence in two dimensions, with intermittency, with thermal convection, in three dimensions with helicity, and in three- and two-dimensional magnetohydrodynamics. The results are summarised in the table 1 together with corresponding results obtained by other means (experimental, numerical, etc). The developments that are needed in each of the cases considered are reported in the appendices. Our results in the case of two-dimensional turbulence and turbulence of convective fluid motions are consistent with the phenomenological theories.

Table 1.

	$E(k)$ by the variational principle	Other results for $E(k)$ from the literature
Two-dimensional turbulence	$E(k)\sim\begin{cases}k^{-5/3} & k\leqslant k_0\\ k^{-3} & k\geqslant k_0\end{cases}$	$E(k)\sim\begin{cases}k^{-5/3} & k\leqslant k_0\\ k^{-3} & k\geqslant k_0\end{cases}$ (Kraichnan 1967, Batchelor 1969)
Intermittency corrections	$E(k)\sim k^{-5/3}P_k^{1/3}$	$E(k)\sim k^{-5/3}P_k^{1/3}$ (Frisch *et al* 1978)
Turbulence of convective fluid motions	$E(k)\sim k^{-11/5}$	$E(k)\sim k^{-11/5}$ (Bolgiano 1950, Obukhov 1959)
Three-dimensional turbulence with helicity	$E(k)\sim\begin{cases}k^{-5/3} & k\leqslant k_0\\ k^{-7/3} & k\geqslant k_0\end{cases}$	$E(k)\sim\begin{cases}k^{-5/3} & k\leqslant k_0\\ k^{-7/3} & k\geqslant k_0\end{cases}$ (Brissaud *et al* 1973) $E(k)\sim k^{-5/3}$ (André and Lesieur 1977)
Three-dimensional magnetohydrodynamic turbulence	$E(k)\sim\begin{cases}k^{-1} & k\leqslant k_0\\ k^{-5/3} & k\geqslant k_0\end{cases}$	$E(k)\sim\begin{cases}k^{-1} & k\leqslant k_0\\ k^{-3/2} & k\geqslant k_0\end{cases}$ (Pouquet *et al* 1976)
Two-dimensional magnetohydrodynamic turbulence	$E(k)\sim\begin{cases}k^{-1/3} & k\leqslant k_0\\ k^{-5/3} & k\geqslant k_0\end{cases}$	$E(k)\sim\begin{cases}k^{-1/3} & k\leqslant k_0\\ k^{-3/2} & k\geqslant k_0\end{cases}$ (Pouquet 1978)

In the case of intermittency correction our result is equal to one obtained by Frisch *et al* (1978) from other considerations.

We remind the reader that the probability P_k of having an intermittent configuration at scale k^{-1} is given by:

$$P_k=(k/k_0)^{-(d-D)}$$

where d is the physical dimension of the space and D is the Mandelbrot 'fractal' dimension (Mandelbrot 1974, 1976). D is determined by the dynamics of the system (Mori 1980) as a result of which P_k strongly depends on the system of equations being examined.

In three-dimensional turbulence with helicity a spectrum identical to our one was obtained by Brissaud *et al* (1973) through phenomenological considerations (helicity cascade for $k \geqslant k_0$ and energy cascade for $k \leqslant k_0$). The hypothesis of a simultaneous energy and helicity cascade for $k \geqslant k_0$ leads to $E(k) \sim k^{-5/3}$. This last result was obtained by André and Lesieur (1977) also with the EDQNM (eddy damped quasi-normal Markovian) approximation. There are however no experimental results.

In three-dimensional magnetohydrodynamic turbulence our results differ from those of Pouquet *et al* (1976) over the range $k \geqslant k_0$ by nearly 10% (in the exponent value); this disagreement may be due to the fact that we consider only local interactions in the k space. However, if a law of the form

$$E(k) \leqslant ck^{-\delta} \tag{3.1}$$

must hold, then we must have (Pouquet 1978)

$$\delta \geqslant \begin{cases} 5/3 & d=3 \\ 4/3 & d=2. \end{cases} \tag{3.2}$$

Our results agree with (3.2) while those of Pouquet *et al* (1976) obtained through the EDQNM approximation do not.

Also in the two-dimensional magnetohydrodynamic case we obtain a perfect agreement between the two approaches in the range $k \leqslant k_0$ and a weak 10% difference in the range $k \geqslant k_0$.

4. Some analogies with statistical mechanics

In this section we want to point out some analogies between statistical mechanics and fully developed turbulence. Our considerations are not an attempt to define a statistical mechanics of turbulence but only an attempt to a better understanding of the behaviour of turbulent flows.

In statistical mechanics the temperature T (the inverse of the Lagrange multiplier β) is connected to energy fluctuation (in a continuous system)

$$\langle H^2 \rangle - \langle H \rangle^2 \sim T^2 \tag{4.1}$$

where H is the energy of the system.

In a turbulent system, where a variational principle is supposed to hold, Λ is the analogue of β while the mean energy dissipation $\langle \varepsilon \rangle$ corresponds to T. It thus seems, at least formally, that a parallelism is established between T and $\langle \varepsilon \rangle$.

This analogy seems to be demonstrated by the fact that for a turbulent system also the equation (4.1) holds changing T into $\langle \varepsilon \rangle$.

Indeed, let us write the NS equations in the Fourier space

$$\partial_t V_\alpha(\boldsymbol{k}) = -\nu k^2 V_\alpha(\boldsymbol{k}) + \sum_{\boldsymbol{k}', \boldsymbol{k}'', \beta, \gamma} A_{\alpha\beta\gamma}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') V_\beta(\boldsymbol{k}') V_\gamma(\boldsymbol{k}'') + F_\alpha(\boldsymbol{k}). \tag{4.2}$$

$F_\alpha(\boldsymbol{k})$ is the forcing term injecting energy on the large scale

$$F_\alpha(\boldsymbol{k}) \simeq F_\alpha \delta(\boldsymbol{k} - \boldsymbol{k}_0) \tag{4.3}$$

while

$$V_\alpha(\boldsymbol{k}) = \delta V_\alpha(\boldsymbol{k}) \qquad \langle V_\alpha(\boldsymbol{k}) \rangle = 0 \qquad \text{for } k \neq k_0 \tag{4.4}$$

$$V_\alpha(\boldsymbol{k}_0) = \langle V_\alpha(\boldsymbol{k}_0) \rangle + \delta V_\alpha(\boldsymbol{k}_0) \qquad \langle \delta V_\alpha(\boldsymbol{k}_0) \rangle = 0. \tag{4.4'}$$

The energy of the system is

$$E = \tfrac{1}{2} \sum_{k,\alpha} |V_\alpha(\boldsymbol{k})|^2 \tag{4.5}$$

giving

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\sum_{\alpha,k} \nu k^2 |V_\alpha(\boldsymbol{k})|^2 + \sum_{\alpha,k} F_\alpha(\boldsymbol{k}) V_\alpha(\boldsymbol{k}). \tag{4.6}$$

At stationary equilibrium there is balance between injection and dissipation and we have

$$\langle \mathrm{d}E/\mathrm{d}t \rangle = 0 \tag{4.7}$$

that is $\langle E \rangle$ = constant where the average $\langle \cdot \rangle$ means average as an ensemble of repetitions (randomness may be introduced for example at the level of the initial conditions).

From (4.6) and (4.7) we obtain

$$\langle \varepsilon \rangle = \sum_{\alpha,k} \nu k^2 \langle |V_\alpha(\boldsymbol{k})|^2 \rangle \sim F_\alpha \langle V_\alpha(\boldsymbol{k}_0) \rangle \tag{4.8}$$

from which

$$\left\langle \left(\frac{\mathrm{d}E}{\mathrm{d}t} \right)^2 \right\rangle = \left\langle \left(-\sum_{\alpha,k} \nu k^2 |V_\alpha(\boldsymbol{k})|^2 + \sum_{\alpha,k} F_\alpha(\boldsymbol{k}) V_\alpha(\boldsymbol{k}) \right)^2 \right\rangle$$

$$\simeq \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2 = \sigma_\varepsilon^2 \tag{4.9}$$

under the very common and reasonable assumption of statistical independence of large scale velocity fluctuation with respect to that of the other scales. On the other hand, experimental data give (Kuo and Corrsin 1971)

$$\langle \varepsilon^2 \rangle = C \langle \varepsilon \rangle^2 \tag{4.10}$$

with $C \simeq 4$ for $d = 3$.

Equations (4.8) and (4.10) imply

$$\langle (\mathrm{d}E/\mathrm{d}t)^2 \rangle \simeq C_1 \langle \varepsilon \rangle^2 \qquad C_1 \simeq 3. \tag{4.11}$$

On the other hand (Monin and Yaglom 1975, p 87) we know that

$$\langle (E(\tau) - E(0))^2 \rangle = \int_0^\infty (1 - \cos \omega\tau) H(\omega)\, \mathrm{d}\omega \tag{4.12}$$

where $H(\omega)$ is the amplitude at frequency ω of $\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2$.

Because the energy is almost entirely contained within the large scales of the motion

$$\sigma_E^2 = \int_0^\infty H(\omega)\, \mathrm{d}\omega \simeq \int_0^{\omega_0} H(\omega)\, \mathrm{d}\omega \tag{4.13}$$

where $\omega_0 \sim T_0^{-1}$, $T_0 = 1/k_0 V_0$.

If $\tau \ll T_0$, it follows from (4.12) that

$$\langle (E(\tau) - E(0))^2 \rangle \simeq \int_0^{\omega_0} \tfrac{1}{2} \tau^2 \omega^2 H(\omega)\, \mathrm{d}\omega$$

$$\simeq \tfrac{1}{2} \tau^2 \omega_0^{*2} \int_0^{\omega_0} H(\omega)\, \mathrm{d}\omega \simeq \tfrac{1}{2} \tau^2 \omega_0^{*2} \sigma_E^2 \tag{4.14}$$

where $\omega_0^* \sim \omega_0$; as $\tau \to 0$ equation (4.14) gives

$$\langle (\mathrm{d}E/\mathrm{d}t)^2 \rangle \sim \omega_0^{*2} \sigma_E^2 \sim T_0^{-2} \sigma_E^2. \tag{4.15}$$

Now, by combining (4.15) and (4.11) we have

$$\langle E^2 \rangle - \langle E \rangle^2 \sim \langle \varepsilon \rangle^2 \tag{4.16}$$

which is the equivalent expression of (4.1) in turbulence. This result has interesting consequences in d-dimensional turbulence with $d \gg 1$.

In d dimensions we have

$$\begin{aligned} \langle \varepsilon \rangle &= \sum_{i=1}^{d} \langle \varepsilon_i \rangle = d \langle \varepsilon_1 \rangle \qquad && \sigma_\varepsilon^2 = d\sigma_{\varepsilon_1}^2 \\ \langle E \rangle &= \sum_{i=1}^{d} \langle E_i \rangle = d \langle E_1 \rangle \qquad && \sigma_E^2 = d\sigma_{E_1}^2 \end{aligned} \tag{4.17}$$

where

$$\varepsilon_i = \nu \sum_{k} k^2 |V_i(\boldsymbol{k})|^2 \qquad E_i = \tfrac{1}{2} \sum |V_i(\boldsymbol{k})|^2$$

are the energy dissipation rate and energy of the ith component ($i = 1, 2, \ldots d$). From (4.14) one derives

$$\langle (E(\tau) - E(0))^2 \rangle \simeq \tfrac{1}{2} \tau^2 \omega_0^*(d)^2 \sigma_E^2 \tag{4.18}$$

where $\omega_0^*(d) \sim 1/T_0(d) \propto d^{-1/2}$ (Fournier *et al* 1978) and by which

$$\langle (\mathrm{d}E/\mathrm{d}t)^2 \rangle \sim \omega_0^*(d)^2 \sigma_E^2 \propto \sigma_E^2 / d. \tag{4.18$'$}$$

Now, using (4.18′), (4.17) and (4.9) we have

$$\sigma_{\varepsilon_1}^2 \sim \omega_0^*(d)^2 \sigma_{E_1}^2 \propto \sigma_{E_1}^2 / d; \tag{4.19}$$

energy dissipation fluctuations are thus negligible when $d \to \infty$ and one may expect from this that the Kolmogorov law is right in the limit $d \to \infty$ corresponding to a mean field theory (Kraichnan 1974, Ma 1976).

5. Conclusion

With this variational principle one obtains, in a unified way, many results previously derived by various arguments. Our hope is that this approach may play the role of a 'mean field theory'.

It is possible that this variational principle corresponds to a saddle point approximation on the 'true' probability distribution. It would be interesting to understand the connection (if any) with the more mathematically minded approach of Ruelle (1980a, b).

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Appendix 1. Two-dimensional turbulence

In two-dimensional turbulence, besides energy the enstrophy is an integral of motion

$$\Omega=\tfrac{1}{2}\int|\omega(\boldsymbol{x})|^2\,\mathrm{d}^2\boldsymbol{x}\qquad\omega(\boldsymbol{x})=\mathrm{rot}\,\boldsymbol{V}(\boldsymbol{x}).\tag{A1.1}$$

For viscous fluid motion the mean value of the enstrophy will be conserved as a result of the balancing between viscous dissipation and external enstrophy input.

By writing $\Omega=\frac{1}{2}\Sigma_n k_n^2V_n^2$ our variational principle becomes

$$\frac{\partial}{\partial V_n}(T_p+\Lambda_1E+\Lambda_2\Omega)=0.\tag{A1.2}$$

Equation (A1.2) implies

$$V_n=\frac{1}{(\Lambda_1k_n+\Lambda_2k_n^3)^{1/3}}.\tag{A1.3}$$

It is easily verified that $\Lambda_1\sim\langle\varepsilon\rangle^{-1}$ and $\Lambda_2\sim\langle\eta\rangle^{-1}$ where η is the enstrophy dissipation rate; by (A1.3) it follows that

$$E(k)\sim\begin{cases}k^{-5/3} & k\leqslant k_0\\ k^{-3} & k\geqslant k_0.\end{cases}\tag{A1.4}$$

Appendix 2. Intermittency corrections

We consider V_n as a bivalued stochastic process (Frisch *et al* 1978, Benzi and Vulpiani 1980); one of these two values, i.e. the one referring to the intermittent configuration, will occur with probability P_n; the other, which refers to the non-intermittent configuration, will occur with probability $(1-P_n)$.

We thus write

$$V_n=\begin{cases}V_n^l\sim0 & \text{'laminar' configuration}\\ V_n^i & \text{intermittent configuration}\end{cases}\tag{A2.1}$$

and thus obtain for τ_n

$$\tau_n=1/k_nV_n^i\tag{A2.2}$$

the flow dynamics being dominated by intermittent fluctuations.

The mean energy of the flow (which is conserved) is

$$E=\tfrac{1}{2}\sum_n P_n(V_n^i)^2;\tag{A2.3}$$

the variational principle implies

$$E(k_n)\sim(V_n^i)^2P_nk_n^{-1}=\Lambda^{-2/3}P_n^{1/3}k_n^{-5/3}.\tag{A2.4}$$

Appendix 3. Turbulence of convective fluid motions

The equations describing this phenomenon are the Boussinesq ones (Chandrasekhar

1961), that is the NS equations plus the heat equation under suitable approximations

$$\begin{aligned} &\partial_t \boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = \nu\Delta\boldsymbol{V} - \rho^{-1}\nabla p + \Gamma T'\boldsymbol{m} \\ &\nabla\cdot\boldsymbol{V} = 0 \\ &\partial_t T' + (\boldsymbol{V}\cdot\nabla)T' = \chi\Delta T' + f \end{aligned} \tag{A3.1}$$

where $\Gamma = g\beta$, g is the gravity acceleration and β is the thermal expansion coefficient; $\boldsymbol{m}$ is the gravity force direction and T' is the temperature derivation from its mean value.

In the Fourier space the nonlinear term in the first equation is of order of $\sim k_n V_n^2$. It is reasonable to assume that, in turbulent conditions, this term has the same order of magnitude as $\Gamma\theta_n$, when θ_n is the corresponding Fourier amplitude of T'. This amounts to assuming that the dynamical evolution of the fields $\boldsymbol{V}$ and T' occurs with the same characteristic times. The following is thus considered to hold

$$\theta_n \sim \Gamma^{-1} V_n^2 k_n. \tag{A3.2}$$

At the stationary state, the forcing term f balances the dissipative one, $\chi\Delta T'$, thus securing the conservation of the mean value of the quantity

$$\int (T')^2\, \mathrm{d}^3\boldsymbol{x}. \tag{A3.3}$$

In k space, by making use of equation (A3.2) the quantity (A3.3) becomes

$$\sum_n \theta_n^2 \sim \Gamma^{-2} \sum_n k_n^2 V_n^4 = \text{constant}. \tag{A3.4}$$

The variational principle implies

$$V_n = \Lambda^{-1/5}\Gamma^{2/5} k_n^{-3/5} \tag{A3.5}$$

from which

$$E(k) \sim \Lambda^{-2/5}\Gamma^{4/5} k^{-11/5}; \tag{A3.6}$$

Λ^{-1} is easily seen to be proportional to $\bar{N}$ (the mean rate of dissipation of the quantity (A3.3)).

Appendix 4. Three-dimensional turbulence with helicity

In a three-dimensional non-viscous fluid also helicity is conserved (as well as energy) (Brissaud *et al* 1973)

$$\mathscr{H} = \int (\boldsymbol{V}\cdot \mathrm{rot}\ \boldsymbol{V})\, \mathrm{d}^3\boldsymbol{x}. \tag{A4.1}$$

For a real viscous fluid, by supplying energy and helicity at a large scale ($\sim l_0$) we may obtain the conservation of their mean values despite dissipation.

The following properties will be conserved

$$E = \tfrac{1}{2}\sum_n V_n^2 \qquad H = \tfrac{1}{2}\sum_n k_n V_n^2. \tag{A4.2}$$

From our principle we obtain

$$V_n = (\Lambda_1 k_n + \Lambda_2 k_n^2)^{-1/3} \tag{A4.3}$$

from which

$$E(k) \sim \begin{cases} k^{-5/3} & k \leqslant k_0 \\ k^{-7/3} & k \geqslant k_0. \end{cases}$$

Appendix 5. Three-dimensional magnetohydrodynamic turbulence

The descriptive equations are ($\boldsymbol{V}$ is the velocity field and $\boldsymbol{b}$ is the magnetic field)

$$\begin{aligned} (\partial_t - \nu\Delta)\boldsymbol{V} &= -(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} + (\boldsymbol{b}\cdot\boldsymbol{\nabla})b - \rho^{-1}\boldsymbol{\nabla}p + f_1 \\ (\partial_t - \lambda\Delta)\boldsymbol{b} &= -(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{b} + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{V} + f_2 \\ \boldsymbol{\nabla}\cdot\boldsymbol{V} &= 0 \\ \boldsymbol{\nabla}\cdot\boldsymbol{b} &= 0. \end{aligned} \tag{A5.1}$$

The following conservation laws (Pouquet *et al* 1976) hold (because of the presence of the forcing terms f_1 and f_2 balancing the dissipative ones) for the mean values of the quantities

$$\tfrac{1}{2}\int (\boldsymbol{V}^2 + \boldsymbol{b}^2)\,\mathrm{d}^3\boldsymbol{x} \qquad \text{energy} \tag{A5.2}$$

$$\tfrac{1}{2}\int (\boldsymbol{a}\cdot\boldsymbol{b})\,\mathrm{d}^3\boldsymbol{x} \qquad \text{magnetic helicity} \tag{A5.3}$$

where $\boldsymbol{b} = \operatorname{rot} \boldsymbol{a}$, that is, $\boldsymbol{a}$ is the magnetic potential. By requiring that coupling terms in $\boldsymbol{b}$ and $\boldsymbol{V}$ in the first two equations of (A5.1) have the same order of magnitude as the transport ones we have

$$b_n \sim V_n \tag{A5.4}$$

and then

$$a_n \sim k_n^{-1} b_n \sim k_n^{-1} V_n. \tag{A5.5}$$

By using (A5.4) and (A5.5), (A5.2) and (A5.3) become

$$E = \tfrac{1}{2}\sum_n V_n^2 \qquad H_{\mathrm{M}} = \tfrac{1}{2}\sum_n k_n^{-1} V_n^2.$$

Our variational principle requires:

$$V_n = (\Lambda_1 k_n + \Lambda_2)^{-1/3} \tag{A5.6}$$

$$E(k) \sim \begin{cases} k^{-1} & k \leqslant k_0 \\ k^{-5/3} & k \geqslant k_0. \end{cases} \tag{A5.7}$$

Appendix 6. Two-dimensional magnetohydrodynamic turbulence

The dynamical equations of the system are the equations (A5.1); while the quantities to be conserved (in the mean) are (Pouquet 1978)

$$\tfrac{1}{2}\int (\boldsymbol{V}^2 + \boldsymbol{b}^2)\,\mathrm{d}^2\boldsymbol{x} \qquad \text{energy} \tag{A6.1}$$

$$\tfrac{1}{2}\int \boldsymbol{a}^2\, \mathrm{d}^2\boldsymbol{x} \qquad \text{square magnetic potential.} \tag{A6.2}$$

The relation $V_n \sim b_n$ holds again, and then

$$E = \tfrac{1}{2}\sum_n V_n^2 = \text{constant} \tag{A6.3}$$

$$A = \tfrac{1}{2}\sum_n k_n^{-2} V_n^2 = \text{constant.} \tag{A6.4}$$

Applying the variational principle we obtain

$$V_n = (\Lambda_1 k_n + \Lambda_2/k_n)^{-1/3}. \tag{A6.5}$$

From (A6.5) we derive

$$E(k) \sim \begin{cases} k^{-1/3} & k \leqslant k_0 \\ k^{-5/3} & k \geqslant k_0. \end{cases} \tag{A6.6}$$

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